

# Linear Time Algorithm for Tree $t$ -spanner in Outerplanar Graphs via Supply-Demand Partition in Trees

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## Abstract

A  $t$ -stretch spanning tree (tree  $t$ -spanner)  $T$  of a graph  $G$  is a spanning tree such that the distance between any two vertices in  $T$  is at most  $t$  times their distance in  $G$ . Given a graph  $G$  and an integer  $t \geq 2$ , the tree  $t$ -spanner problem decides the existence of a tree  $t$ -spanner in  $G$ . We solve the tree  $t$ -spanner in outerplanar graphs with the introduction of the  $S$ -partition problem in graphs which is defined as follows: given a graph  $G$ , a weight function  $w : V(G) \rightarrow \mathbb{N}$ , a set  $S \subseteq V(G)$  of special vertices, and an integer  $t$ , decide whether there exists a partition of  $V(G)$  into  $V_1, \dots, V_{|S|}$  such that for  $1 \leq i \leq |S|$ ,  $G[V_i]$  is connected,  $V_i$  contains exactly one vertex from  $S$  and cost of  $V_i$ , which is the sum of weights of vertices in  $V_i$ , is at most  $t$ . We then present a linear-time reduction from tree  $t$ -spanner in outerplanar graphs to supply-demand partition in trees by using  $S$ -partition in trees as an intermediate problem. As a consequence, we obtain a linear-time algorithm for tree  $t$ -spanner in outerplanar graphs.

## 1 Introduction

*Tree spanners:* Finding a *minimum stretch spanning tree* is a classical optimization problem in algorithmic graph theory that has several applications in networks, distributed systems and biology [1, 16, 2]. A  $t$ -stretch spanning tree (also known as tree  $t$ -spanner) of an unweighted graph  $G$  is a spanning tree  $T$  such that for every two vertices the distance in  $T$  is at most  $t$  times their distance in  $G$ . Here, the parameter  $t$  is called *stretch* and this notion was introduced by Peleg and Ullman [18]. A *minimum stretch spanning tree (MSST)* of  $G$  is a  $t$ -stretch spanning tree of  $G$  in which  $t$  is minimum. The problem of determining whether a graph  $G$  contains a tree  $t$ -spanner is the *tree  $t$ -spanner* problem and the MSST problem is the corresponding optimization version.

*Past results on tree spanners:* As far as complexity is concerned, Cai and Corneil in [6] showed that *tree  $t$ -spanner* is NP-complete for any  $t \geq 4$  and polynomial-time solvable for  $t \leq 2$ . The status of the problem for  $t = 3$  is open. Since tree  $t$ -spanner is NP-complete in general graphs, the problem of tree  $t$ -spanner in special classes of graphs has attracted many researchers in the literature. In particular, the study on tree spanners in planar graphs was initiated by Fekete and Kremer [9], who showed that tree  $t$ -spanner in planar graphs is NP-complete when  $t$  is a part of the input and polynomial-time solvable if  $t \leq 3$ . Later, Fomin et al. showed that, tree  $t$ -spanner problem parameterized by  $t$  is fixed parameter tractable in planar graphs (also in apex-minor free graphs) [8]. Also, for any fixed  $t \geq 4$ , the tree  $t$ -spanner is NP-complete even on chordal graphs [4] and chordal bipartite graphs [5]. However, interval graphs, permutation graphs, regular bipartite graphs and distance hereditary graphs admit a tree 3-spanner [12, 14] that can be found in polynomial time. As a consequence, MSST in such graphs can be found in polynomial time due to the fact that tree 2-spanner in general graphs is polynomial-time decidable.

**Our contribution:** An overview of the results presented in this paper is given below. We formally define and prove the results in appropriate sections.

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1. (*In Section 3.1*) To decide the existence of a tree  $t$ -spanner in outerplanar graphs, we come up with a new problem called  $S$ -partition (defined below). It turns out that the  $S$ -partition is a variant of the bounded component spanning forest problem (ND10 in [10]). Further, we give a linear-time reduction from tree spanner problem in outerplanar graphs to  $S$ -partition problem in trees. To the best of our knowledge, there is no literature on reduction from the tree spanner problem in outerplanar graphs to a partition problem in trees. For an outerplanar graph  $G$ , the transformation basically produces a weak dual  $\tilde{T}$  of  $G$ . The essence of the reduction is to transform the complexity of the tree spanner problem in outerplanar graphs to a partition problem in trees. In particular, we prove that  $G$  admits a spanning tree with stretch at most  $t$  if and only if  $\tilde{T}$  has a  $S$ -partition with cost at most  $t - 1$ .
2. (*In Section 3.2*) We then obtain a linear-time reduction from  $S$ -partition in trees to the well studied supply-demand tree partition (defined below) [11]. Finally, by using the algorithm in [11] to decide supply-demand tree partition problem, we solve tree  $t$ -spanner in linear time and subsequently MSST in  $O(|V(G)| \log |V(G)|)$  time in outerplanar graphs. Also, we briefly discuss the advantages of our approach when compared with tree decomposition based techniques to solve the tree  $t$ -spanner problem in outerplanar graphs.

*Graph partition problems:* In our work, we introduce the  $S$ -partition problem, a variant of the *bounded component spanning forest* (ND10 in [10]), and relate it to tree spanners in outerplanar graphs. The bounded component spanning forest problem is known to be NP-complete [10]. To the best of our knowledge, there is no literature on  $S$ -partition which we defined as follows.

Tree  $S$ -partition

**Instance:** A tree  $T$ , a weight function  $w : V(T) \rightarrow \mathbb{N}$ , a set  $S \subseteq V(T)$  of special vertices, and  $t \in \mathbb{N}$ .

**Question:** Does there exist a partition of  $V(T)$  into  $|S|$  disjoint sets  $V_1, \dots, V_{|S|}$  such that for  $1 \leq i \leq |S|$ ,  $T[V_i]$  is connected,  $|V_i \cap S| = 1$  and  $\sum_{v \in V_i} w(v) \leq t$  ?

We then relate  $S$ -partition in trees to *supply-demand* partition in trees.

Supply-demand tree partition [11]

**Instance:** A Tree  $T$  such that  $V(T) = V_s \uplus V_d$ , a supply function  $s : V_s \rightarrow \mathbb{R}^+$ , and a demand function  $d : V_d \rightarrow \mathbb{R}^+$ .

**Question:** Does there exist a partition of  $V(T)$  into disjoint sets  $V_1, \dots, V_k$ , where  $k = |V_s|$ ,  $V_i$  contains exactly one vertex  $u \in V_s$  and  $\sum_{v \in V_i \setminus V_s} d(v) \leq s(u)$  ?

## 2 Preliminaries

Through out this paper, we consider tree spanners in outerplanar graphs.

*Graph theoretic preliminaries:* In this paper, we consider only simple, finite, connected, undirected and unweighted graphs. Let  $G = (V, E)$  be a graph where  $V(G)$  is the set of vertices and  $E(G) \subseteq \{(u, v) \mid u, v \in V(G), u \neq v\}$  is the set of edges. Order of  $G$  and size of  $G$  are  $|V(G)|$  and  $|E(G)|$ , respectively. For a vertex  $v \in V(G)$ , the *neighborhood* of  $v$  is the set  $\{u \mid (u, v) \in E(G)\}$  and is denoted by  $N_G(v)$ . The degree of a vertex  $v \in V(G)$ ,  $\deg_G(v)$  is  $|N_G(v)|$ . For a set  $S \subseteq V(G)$ ,  $G[S]$  denotes the graph induced on the set  $S$  and  $G \setminus S$  denotes  $G[V(G) \setminus S]$ . A *vertex cut* of a graph  $G$  is a set  $S \subseteq V(G)$  such that  $G \setminus S$  has more than one component. The *connectivity* of  $G$  is the minimum size of a vertex set  $S$  such that  $G \setminus S$  is disconnected or has only one vertex. The edge  $e$  between the vertices  $u$  and  $v$  is denoted by  $e(u, v)$ .  $V_1, \dots, V_r$  is a partition of a set  $V$  if and only if  $\bigcup_{i=1}^r V_i = V$ , and  $\forall i, j$   $1 \leq i < j \leq r$ ,  $V_i \cap V_j = \emptyset$ . Each such  $V_i$  is referred to as a part of the partition.

*Tree spanner preliminaries:* The number of edges in a shortest path from vertex  $u$  to vertex  $v$  in  $G$  is called the *distance* between  $u$  and  $v$ , and is denoted by  $d_G(u, v)$ . A spanning tree  $T$  of a graph  $G$  is said to be a *tree  $t$ -spanner* if the distance between any two vertices in  $T$  is at most  $t$  times their distance in  $G$ . We refer to  $t$  as the *stretch* of  $T$ . A graph that has a *tree  $t$ -spanner* is called a *tree  $t$ -spanner admissible graph*. From [6], it is known that in any spanning tree  $T$  of  $G$ ,  $d_T(u, v) \leq t \cdot d_G(u, v)$  for every pair  $(u, v) \in V(G) \times V(G)$  if and only if  $d_T(u, v) \leq t$  for every edge  $(u, v) \in E(G) \setminus E(T)$ . For any edge  $e(u, v) \in E(G) \setminus E(T)$ , the stretch of  $e$  is defined as the distance between  $u$  and  $v$  in  $T$ . Stretch of  $e$  is denoted by  $Stretch(e)$ . If  $e \in E(T)$  then  $Stretch(e) = 1$ , otherwise  $Stretch(e) = d_T(u, v)$ . Stretch of spanning tree  $T$  is  $Stretch(T) = \max\{Stretch(e) \mid e \in E(G) \setminus E(T)\}$ . A spanning tree of  $G$ , with minimum stretch is referred to as a *minimum stretch spanning tree (MSST)* of  $G$ . The stretch of MSST of  $G$  is denoted by  $Stretch(G)$ . We use the terms *tree  $t$ -spanner* and *spanning tree with stretch at most  $t$* , interchangeably.

*Planar graph preliminaries:* A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their ends; otherwise it is *nonplanar*. A planar graph is *outerplanar* if it can be embedded on the plane such that all vertices lie on the boundary of its exterior region and such an embedding is called an *outerplanar embedding*. In this paper, a planar graph and an outerplanar graph always refer to a fixed planar and outerplanar embedding, respectively. The regions defined by the planar embedding of a planar graph  $G$  are *faces* of  $G$  and set of all faces of  $G$  are denoted by  $Faces(G)$ . For each face  $f \in Faces(G)$ ,  $V(f)$  and  $E(f)$ , denote the set of vertices and set of edges of  $f$ , respectively. Two faces  $f, f' \in Faces(G)$  are *adjacent* if  $E(f) \cap E(f') \neq \emptyset$ . In every planar embedding, there is one unbounded face called *exterior* face and it is denoted by  $f_{ext}$ . An edge  $e$  is *external* if  $e \in E(f_{ext})$ , otherwise it is *internal*. All the bounded faces of  $G$  are *interior*. Interior faces that are adjacent to the exterior face are called *E-faces* and other interior faces are called *I-faces*. The set of interior faces, *I-faces* and *E-faces* of  $G$  are denoted by  $Int-faces(G)$ ,  $I-faces(G)$ , and  $E-faces(G)$ , respectively. Also,  $Faces(G) = \{f_{ext}\} \cup Int-faces(G)$  and  $Int-faces(G) = E-faces(G) \cup I-faces(G)$ .

### 3 Tree Spanners via Supply-Demand Tree Partition

We now present a few observations and some key lemmas on outerplanar graphs and structural properties of tree spanners in outerplanar graphs.

**Lemma 3.1** [6] *A spanning tree  $T$  of a graph  $G$  is a tree  $t$ -spanner if and only if  $d_T(x, y) \leq t$  for every edge  $(x, y) \in E(G) \setminus E(T)$ .*

**Lemma 3.2** [13] *An outerplanar graph is Hamiltonian if and only if it is 2-connected. Moreover every 2-connected outerplanar graph contains a unique Hamiltonian cycle.*

It is easy to see that the vertex connectivity of any outerplanar graph is at most two. If an outerplanar graph  $G$  has cut vertices, a minimum stretch spanning tree of  $G$  can be obtained by performing a union on minimum stretch spanning trees of the maximal vertex 2-connected components of  $G$ . So without loss of generality, we consider vertex 2-connected outerplanar graphs. For subsequent discussions, let  $G$  denote a vertex 2-connected outerplanar graph. We refer to the internal and external edges with respect to the graph  $G$ . The same holds for *E-faces*, *I-faces* and *Int-faces*. For a spanning tree  $T$  of  $G$ ,  $P_T(u, v)$  denotes the path between vertices  $u$  and  $v$  in  $T$ .

**Lemma 3.3** [17] *Let  $T$  be a spanning tree of  $G$ . Then there exists an external edge  $e(x, y) \in E(G) \setminus E(T)$  such that  $Stretch(T) = d_T(x, y)$ .*

The structure of a *canonical tree  $t$ -spanner* in outerplanar graphs is shown below.

**Lemma 3.4** *Let  $G$  admit a tree  $t$ -spanner. Then there exists a tree  $t$ -spanner  $T$  of  $G$  that satisfies the following:*

1. For every  $E$ -face  $f$  of  $G$ , there exists exactly one external edge  $e(u, v) \in E(f)$  such that  $e \notin E(T)$ .
2. For every external edge  $e(u, v) \notin E(T)$ , there exists exactly one  $E$ -face of  $G$  in  $G[V(C)]$ , where  $C = P_T(u, v) \cup \{e(u, v)\}$ .

**Proof** Let  $T'$  be a tree  $t$ -spanner of  $G$ . Claim 1 follows easily if  $|Int-faces(G)| = 1$ . So consider the case where  $|Int-faces(G)| \geq 2$ . Let  $f \in E-faces(G)$ . If two or more external edges of  $f$  are not present in  $T'$ , then  $T'$  is not a tree and leads to a contradiction that  $T'$  is a tree. Suppose there exists a face  $f \in E-faces(G)$ , such that all the external edges in  $E(f)$  are present in  $E(T')$ . Then definitely there exists an internal edge  $e \in E(f) \setminus E(T')$ , because  $T'$  is a tree. Let  $e' \in E(f)$  be an external edge. Now  $T' = T' \cup \{e\} \setminus \{e'\}$  is a spanning tree of  $G$ . Observe that adding an internal edge of  $f$  to  $T'$  and removing an external edge of  $f$  from  $T'$  do not increase the stretch of  $T'$ . By repeated application of the above process for at most  $|E-faces(G)|$  times, we obtain a tree  $t$ -spanner  $T$  that satisfies Claim 1. We now prove that  $T$  satisfies Claim 2. Let  $e(u, v) \in E(G) \setminus E(T)$  be an external edge. Let  $C_e = P_T(u, v) \cup \{e(u, v)\}$  and  $H = G[V(C_e)]$ . If  $H$  contains no  $E$ -face of  $G$ , then it implies that  $e$  is an internal edge, a contradiction to the premise that  $e$  is an external edge. Assume that  $H$  contains at least two  $E$ -faces  $f$  and  $f'$  of  $G$ . Without loss of generality, let  $E(f)$  contain  $e$ . Then all the external edges of  $f'$  are in  $T$ , a contradiction to Claim 1.  $\square$

A tree  $t$ -spanner of  $G$  that satisfies Claim 1 and Claim 2 of Lemma 3.4 is called a *canonical tree  $t$ -spanner*. From Lemma 3.4, it is clear that the number of external edges missing in a canonical tree  $t$ -spanner of  $G$  is equal to the number of  $E$ -faces of  $G$ . When the lengths of all interior faces of an outerplanar graph are known, the following lemma gives the length of the Hamiltonian cycle in terms of length of interior faces. This lemma is crucially used in Section 3.1.

**Lemma 3.5** *In an outerplanar graph  $G$  with interior face lengths  $l_1, \dots, l_r$  and the Hamiltonian cycle  $C$ ,  $|C| = (\sum_{i=1}^r l_i) - 2(r - 1)$ .*

**Proof** Let  $l_{ext} = |V(f_{ext})|$ . Since  $2|E(G)| = l_1 + \dots + l_r + l_{ext}$ , by Euler's planarity formula, we obtain  $|C| = |V(G)| = |E(G)| - r + 1 = 2|E(G)| - 2r + 2 - |V(G)| = l_1 + \dots + l_r - 2(r - 1) + l_{ext} - |V(G)|$ . Since  $G$  is an outerplanar,  $l_{ext} = |V(G)|$ , thereby  $|C| = (\sum_{i=1}^r l_i) - 2(r - 1)$ .  $\square$

### 3.1 Tree spanners via tree $S$ -partition

We relate the problem of tree  $t$ -spanners in outerplanar graphs to the problem of  $S$ -partition in trees. We now recall the  $S$ -partition problem in trees. Let  $T$  be a tree,  $w$  be a weight function  $w : V(T) \rightarrow \mathbb{N}$ ,  $t$  be a positive integer and  $S \subseteq V(T)$  be a set of *special* vertices. An  $S$ -partition  $\pi(V(T))$  of  $T$  is a partition of  $V(T)$  into parts  $V_1, V_2, \dots, V_{|S|}$  such that for  $1 \leq i \leq |S|$ ,  $T[V_i]$  is connected and  $|V_i \cap S| = 1$ . For a subset  $X \subseteq V$ ,  $Cost(X) = \sum_{v \in X} w(v)$  and cost of the partition  $Cost(\pi(V(T))) = \max\{Cost(V_i) \mid V_i \in \pi(V(T))\}$ . By  $t$ - $S$ -partition of  $T$ , we mean an  $S$ -partition  $\pi$  of  $V(T)$ , such that  $Cost(\pi) \leq t$ . The two distinct parts  $V_i$  and  $V_j$  of  $V(T)$  are *adjacent* if there is an edge  $e(u, v) \in E(T)$  such that  $u \in V_i$  and  $v \in V_j$ . The decision version of  $S$ -partition problem is formulated as follows:

**Instance:** A tree  $T$ , a weight function  $w : V(T) \rightarrow \mathbb{N}$ , a positive integer  $t$  and a set  $S \subseteq V(T)$  of special vertices.

**Question:** Does there exist an  $S$ -partition  $\pi$  for  $T$ , such that  $Cost(\pi) \leq t$ ?

**Reduction from tree spanner in outerplanar graphs to  $S$ -partition in trees:** We now present a linear-time reduction from tree  $t$ -spanner in outerplanar graphs to  $S$ -partition in trees with cost at most  $(t - 1)$ . For a given outerplanar graph  $G$ , the output instance  $\tilde{T}$  along with a set of special vertices  $S \subseteq V(\tilde{T})$  and a weight function  $w : V(\tilde{T}) \rightarrow \mathbb{N}$  is constructed as follows:  $V(\tilde{T}) = \{v_f \mid f \in Int-faces(G)\}$ ,  $E(\tilde{T}) = \{(v_f, v_g) \mid f, g \in Int-faces(G), |E(f) \cap E(g)| = 1\}$ . The weight function  $w : V(\tilde{T}) \rightarrow \mathbb{N}$  is such that  $\forall v_f \in V(\tilde{T}), w(v_f) = |E(f)| - 2$ . The set of special vertices  $S$  is  $\{v_f \mid f \in E-faces(G)\}$ . Since  $G$  is 2-connected outerplanar graph, from the above construction  $\tilde{T}$  is a tree. For clarity,  $\tilde{T}$  is weak dual of  $G$ . We now observe some structural properties of outerplanar graphs and then prove the described reduction.

**Lemma 3.6** *Let  $C$  be a cycle in  $G$ ,  $\hat{F}(C) = \text{Int-faces}(G[V(C)])$ ,  $V' = \{v_f \in V(\tilde{T}) \mid f \in \hat{F}(C)\}$ . Then  $\text{Cost}(V') = |C| - 2$ .*

**Proof** Let  $\hat{F}(C) = \{f_1, \dots, f_r\}$  and  $l_i = |E(f_i)|$ . By definition,  $|V'| = r$ . From the transformation, we obtain  $\text{Cost}(V') = \sum_{v \in V'} w(v) = \sum_{i=1}^r (l_i - 2) = \sum_{i=1}^r l_i - 2(r - 1) - 2$ . By Lemma 3.5, we have  $\sum_{i=1}^r l_i - 2(r - 1) - 2 = |C| - 2$ . Hence,  $\text{Cost}(V') = |C| - 2$ .  $\square$

**Observation 1** *Let  $F' \subseteq \text{Int-faces}(G)$ ,  $X = \bigcup_{f \in F'} V(f)$ . Let  $V' = \{v_f \mid f \in F'\}$ . The outerplanar graph  $G[X]$  is 2-connected if and only if  $\tilde{T}[V']$  is connected.*

From the transformation and the following theorem, deciding the existence of tree  $t$ -spanner in outerplanar graphs can be done in linear time if we obtain a linear-time algorithm for  $S$ -partition in trees.

**Theorem 3.7**  *$G$  admits a tree  $t$ -spanner iff  $\tilde{T}$  has a  $(t - 1)$ - $S$ -partition.*

**Proof Necessity.** Let  $T$  be a canonical tree  $t$ -spanner of  $G$ . For an external edge  $e(u, v)$  such that  $e \in E(G) \setminus E(T)$ , we construct a part  $V_e \in \pi(V(\tilde{T}))$  with the desired properties as follows. Consider the cycle  $C = P_T(u, v) \cup \{e(u, v)\}$  in  $G$ . Clearly,  $|C| \leq t + 1$  because  $|P_T(u, v)| \leq t$ . Note that  $G_e = G[V(C)]$  is a 2-connected outerplanar graph. Let  $V_e = \{v_f \mid f \in \text{Int-faces}(G_e)\}$ . Now  $V_e$  is a part in  $\pi(V(\tilde{T}))$ . By Lemma 3.4, there is exactly one face  $f$  in  $G_e$  such that  $f$  is an E-face in  $G$ . By transformation,  $v_f \in V_e$  and since  $v_f \in S$ ,  $|V_e \cap S| = 1$ . From Lemma 3.6,  $\text{cost}(V_e) = |C| - 2$ . So,  $\text{cost}(V_e) \leq t - 1$ . Moreover it is easy to see that  $T[V_e]$  is connected from Observation 1. For each external edge  $e \in E(G) \setminus E(T)$ , a part can be constructed with desired properties using the same approach. From Lemma 3.4, the number of parts in  $\pi(V(\tilde{T}))$  is  $|S|$ . Hence  $\tilde{T}$  has a  $(t - 1)$ - $S$ -partition.

**Sufficiency.** Let  $\pi = \{V_1, V_2, \dots, V_q\}$  be a  $(t - 1)$ - $S$ -partition of  $V(\tilde{T})$ , where  $|S| = q$ . From  $\pi$ , we present a construction of tree  $t$ -spanner for  $G$ . Consider a part  $V_i \in \pi$  for some  $i \in \{1, \dots, q\}$ . Let  $V_i^* = \bigcup_{v_f \in V_i} V(f)$ . Since  $V_i$  is connected,  $G_i = G[V_i^*]$  is connected. Moreover  $G_i$  is 2-connected outerplanar from Observation 1. Note that  $\text{cost}(V_i) \leq t - 1$ , because  $V_i \in \pi(V(\tilde{T}))$ . From Lemma 3.2,  $G_i$  contains the Hamiltonian cycle  $C_i$ . From Lemma 3.6,  $|C_i| = \text{cost}(V_i) + 2$ , so  $|C_i| \leq t + 1$ . Since  $|V_i \cap S| = 1$ ,  $G_i$  contains exactly one E-face of  $G$ . Let  $e_i$  be an external edge of  $G$  such that  $e_i \in E(G_i)$ . The spanning tree  $T_i$  of  $G_i$  such that  $E(T_i) = E(C_i) \setminus \{e_i\}$  is a Hamiltonian path of  $G_i$  and  $\text{Stretch}(T_i) \leq t$ . For each  $G_i, 1 \leq i \leq q$ , a spanning tree  $T_i$  can be obtained in the same way. It is important to observe that in  $\tilde{T}$ , between any two distinct adjacent parts  $V_i$  and  $V_j$  exactly one edge is available. Therefore there is exactly one common edge present in  $T_i$  and  $T_j$ . Since the union of all  $V_i \in \pi$  forms  $V(\tilde{T})$ , the union of all  $T_i$  ( $1 \leq i \leq q$ ) together form the spanning tree  $T$  of  $G$ . For every edge  $(x, y) \in E(G) \setminus E(T)$ ,  $(x, y) \in E(G_i)$  for some  $i \in \{1, \dots, q\}$ , thereby  $\text{Stretch}(e(x, y)) \leq t$ . From Lemma 3.1, it follows that  $T$  is a tree  $t$ -spanner for  $G$ .  $\square$

We now analyze the time complexity of the transformation. The set of interior faces  $\text{Int-faces}(G)$  is required to obtain the weak dual  $\tilde{T}$ . We use Bodlaender's algorithm [3] to obtain an optimal tree decomposition  $\mathcal{T}$  of an outerplanar graph  $G$  and then  $\text{Int-faces}(G)$  from  $\mathcal{T}$ . A tree-decomposition of a graph  $G$  is a pair  $(\{B_i \mid i \in I\}, T)$  where each  $B_i \subseteq V(G)$ , called a bag, and  $T$  is a tree with the elements of  $I$  as nodes. The following three properties must hold:  $\bigcup_{i \in I} B_i = V$ ;  $(u, v) \in E \Rightarrow \exists i \in I$  with  $u, v \in B_i$ ;  $\forall v \in V, \{i \in I \mid v \in B_i\}$  induces a connected subtree of  $T$ . The treewidth of a tree-decomposition  $(\{B_i \mid i \in I\}, T) = \max\{|B_i| \mid i \in I\} - 1$ . The treewidth of a graph  $G$  is the minimum treewidth over all possible tree-decompositions of  $G$  and such tree-decomposition is *optimal tree-decomposition*.

Let  $\mathcal{T}$  be an optimal tree decomposition of  $G$ . For every node  $u \in V(\mathcal{T})$ , bag  $B_u$  denotes the set of vertices associated with node  $u$ . Since outerplanar graphs are of treewidth two,  $\forall u \in V(\mathcal{T}), |B_u| \leq 3$ . Let  $\mathcal{T}'$  be the resultant forest obtained from  $\mathcal{T}$  after deleting all the edges  $e = (u, v) \in E(\mathcal{T})$  for which  $B_u \cap B_v$  forms an edge in  $G$ . For each component  $H$  in  $\mathcal{T}'$ , consider  $X = \bigcup_{u \in V(H)} B_u$ ,  $G[X]$  forms a face in  $G$ . Note that vertices of every internal edge in an outerplanar graph forms a vertex separator. It follows that two faces in  $\text{Int-faces}(G)$  share an edge  $e$  if and only if vertices of edge  $e$  belongs to  $B_u \cap B_v$  for some edge  $(u, v) \in E(\mathcal{T})$ . The construction of  $\mathcal{T}$  [3] and then finding  $\text{Int-faces}(G)$  from  $\mathcal{T}$  take linear time. Also, the

time required for generating the instance  $\langle \tilde{T}, w, S \rangle$  is  $O(|V(G)| + |E(G)|)$ . Since  $|E(G)| \leq 2|V(G)| - 3$  in outerplanar graphs, the transformation takes  $O(|V(G)|)$  time.

Having presented the reduction, it is natural to analyze the complexity of  $S$ -partition problem in trees. In Section 3.2, we reduce  $S$ -partition in trees to supply-demand partition in trees [11] and use the algorithm of supply-demand tree partition to solve tree  $t$ -spanner in outerplanar graphs.

### 3.2 Tree $S$ -partition via supply-demand tree partition

The *supply-demand tree partition* problem [11] is defined as follows. Let  $T$  be a tree in which each vertex is either a supply vertex or a demand vertex. We are also given a positive real number, *supply*, to each supply vertex and *demand* to each demand vertex. We wish to partition the vertices of  $T$  into parts so that each part is a subtree containing exactly one supply vertex and the sum of demands in each subtree is at most the supply of the supply vertex contained in the subtree. Recall that the supply-demand tree partition is a decision problem which is formulated as follows:

#### Supply-Demand Tree Partition

**Instance:** A Tree  $T$  such that  $V(T) = V_s \uplus V_d$ , a supply function  $s : V_s \rightarrow \mathbb{R}^+$ , and a demand function  $d : V_d \rightarrow \mathbb{R}^+$ .

**Question:** Does there exist a partition of  $V(T)$  into disjoint sets  $V_1, \dots, V_k$ , where  $k = |V_s|$ , such that for  $1 \leq i \leq k$ ,  $T[V_i]$  is connected,  $V_i$  contains exactly one vertex  $u \in V_s$  and  $\sum_{v \in V_i \setminus V_s} d(v) \leq s(u)$  ?

We now relate tree  $S$ -partition to supply-demand tree partition by providing a linear-time reduction. Let  $\langle T, S \subseteq V(T), w, t \rangle$  be an input instance of tree  $S$ -partition, where  $T$  is a tree,  $S$  is a set of special vertices of  $T$ ,  $w$  is a weight function and  $t$  is a positive integer. We now describe the construction of  $\langle T', s : V_s(T') \rightarrow \mathbb{R}, c : V_d(T') \rightarrow \mathbb{R} \rangle$  from  $\langle T, S \subseteq V(T), w, t \rangle$ , where  $T'$  is a tree,  $s$  and  $c$  are supply and demand functions, respectively.  $V(T') = V_s(T') \cup V_d(T')$ ,  $V_s = \{u' \mid u \in S\}$ ,  $V_d = V(T) \setminus S$ ,  $E(T') = E(T) \cup \{(u, u') \mid u \in S\}$ . Further, for each  $u \in V_s$ ,  $s(u) = t$  and for each  $u \in V_d$ ,  $d(u) = w(u)$ . Clearly, the reduction takes  $O(|V(T)|)$  time and hence we have the following theorem.

**Theorem 3.8**  $\langle T, S \subseteq V(T), w, t \rangle$  is a YES instance for tree  $S$ -partition if and only if  $\langle T', s : V_s(T') \rightarrow \mathbb{R}, c : V_d(T') \rightarrow \mathbb{R} \rangle$  is a YES instance for supply-demand tree partition.

**Theorem 3.9** Given an outerplanar graph  $G$  on  $n$  vertices and a positive integer  $t$ . Deciding whether  $G$  admits a tree  $t$ -spanner takes  $O(n)$  time.

**Proof** Let  $\langle G, t \rangle$  be an instance of the tree  $t$ -spanner problem in outerplanar graphs. By applying the transformation presented in Section 3.1 on  $\langle G, t \rangle$ , the instance of tree  $S$ -partition  $\langle \tilde{T}, w, S, t - 1 \rangle$  can be obtained in  $O(n)$  time. Also, the instance  $\langle T', s, d \rangle$  of supply-demand tree partition can be obtained in  $O(|V(\tilde{T})|)$  time. By using the algorithm present in [11], the instance of supply-demand tree partition  $\langle T', s, d \rangle$ , can be solved in  $O(|V(T')|)$  time. Therefore, deciding the existence of tree  $t$ -spanner in an outerplanar graph takes  $O(n)$  time.  $\square$

All the proofs we have presented are constructive. Also two of the reductions that we presented and the algorithm used for solving supply-demand tree partition take linear time. So, tree  $t$ -spanner in outerplanar graphs can be constructed in  $O(n)$  time if it exists. Also, from Theorem 3.9, MSST in outerplanar graphs can be constructed in  $O(n \log n)$  time by using binary search on the value of  $t$ .

**Comparison with other approaches to tree  $t$ -spanner:** There are the two standard approaches to solve tree  $t$ -spanner problem in outerplanar graphs. For outerplanar graphs, a tree decomposition of treewidth

at most two can be obtained in linear time using [15, 3]. For any fixed  $t$ , the tree  $t$ -spanner problem can be described as a formula in Monadic Second Order Logic (MSOL) [8]. Given a graph  $G$  and its tree decomposition of treewidth at most  $k$ , Courcelle has established that every graph problem that can be described in MSOL can be solved in  $f(k, l)(|V(G)| + |E(G)|)$ , where  $l$  is the length of the MSOL formula and  $f$  is an arbitrary function that depends only on  $l$  and  $k$  [7]. The above three steps together yields an algorithm for tree  $t$ -spanner problem in outerplanar graphs, that runs in  $O(|V(G)|)$  time when  $t$  is fixed. Clearly, this approach can be used to quickly classify the computational complexity status of the tree  $t$ -spanner problem. Another approach to solve the tree  $t$ -spanner is to apply dynamic programming on an optimal tree decomposition of  $G$ . The first approach is not practical, because the running time of this algorithm depends on the length of the MSOL formula and it results in a large hidden constant factor in the  $O$ -notation. Also, the first approach yields a linear-time algorithm only when  $t$  is fixed. The second approach, dynamic programming demands table maintenance. Also, these approaches do not exploit the structural properties of outerplanar embedding as they work for all graphs with tree width at most two.

Our main contribution is that we have reduced a spanning tree search question in outerplanar graphs to a tree partition question. Subsequently, an algorithm for tree  $t$ -spanner in outerplanar graphs is obtained by an algorithm for the partition problem on trees. This naturally completes the approach of Peleg and Tendler [17] who initiated the study of using the structure of outerplanar embeddings to find MSST. Also, our approach gives the first linear time algorithm for deciding the existence of tree  $t$ -spanner for all values of  $t$  and obtains a tree  $t$ -spanner if it exists. One of the main offshoots of this question is whether this approach can be tailored to other graph classes. Our approach is interesting in the sense that it exploits the properties of outerplanar embeddings.

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